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Comparative Analytical Study of Exact Traveling Wave Solutions in Nonlinear Evolution Equations via Generalized Kudryashov and Modified Simple Equation Methods

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Abstract

In this research, we primarily investigate two distinct methods for determining exact travelling wave solutions: The Modified Simple Equation (MSE) method and The Generalized Kudryashov method (GKM). Using the MSE method, we determined the exact travelling wave solution of the coupled (1+1) dimensional Broer-Kaup (BK) equation. Conversely, the generalized Kudryashov method was employed to find the exact travelling wave solution of the omnipotent unsteady Kortweg-de Vries (KdV) equation and its alternative, the Regularized Long Wave (RLW) equation. The solutions derived are constructed in three distinct classes: hyperbolic function, trigonometric function, and rational function structures. The accuracy of these solutions was verified with mathematical software like MAPLE. The results obtained confirm that both MSE and GKM are highly effective and valuable mathematical tools for solving various types of nonlinear evolution equations (NLEEs) in mathematical physic.

Keywords: *Generalized Kudryashov Method (GKM); Modified Simple Equation Method (MSE); Nonlinear Evolution Equations (NLEEs); Exact Traveling Wave Solutions; Kortweg-de Vries (KdV) Equation; Regularized Long Wave (RLW) Equation; Coupled (1+1) dimensional Broer-Kaup (BK) Equation.*

1. Introduction

Nonlinear evolution equations (NLEEs) play a crucial role in describing complex physical phenomena in fluid mechanics, plasma physics, optical fibers, solid-state physics, and biology [1-27]. Finding exact solutions of these equations is essential to understand the underlying physical mechanisms and dynamic behaviors. Several analytical methods have been developed for this purpose, including the tanh-function [8, 19], extended tanh-function, sine–cosine [20], Jacobi elliptic function [10, 23], F-expansion [1, 22], exp–function [11], and Q-function methods [13].

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Among them, the Q-function or Kudryashov method [28–33] is a powerful algebraic approach for constructing exact traveling wave solutions of high-order nonlinear equations. Another effective approach is the modified simple equation (MSE) method [34–40], introduced by Vitanov, which assumes that the solution can be expressed as a polynomial in an auxiliary function satisfying simpler equations like Bernoulli or Riccati.

In this work, the MSE method is applied to obtain exact analytical solutions of a second-order nonlinear ordinary differential equation (ODE). The results highlight the method's efficiency, simplicity, and capability in solving complex nonlinear problems with high accuracy.

2. Preliminary

2.1 Wave

Wave is a pattern which generated in a medium when a disturbance (caring energy) travel from one point to another point with no transportation of particles that means particles only vibrates about their mean position and only its energy which is transported.

Example: Sound wave (only sound energy transfer), water wave (only water energy transfers), string wave, spring wave etc.

2.2 Travelling wave

When a wave is start from a point in a medium and propagates in all possible direction and never returns, then this type of wave is called travelling wave. It is also called progressive wave. In this wave, the wave amplitude is same for all the elements in the medium and the amplitude is not dependent on the position of the elements of that medium.

Example: Water wave: if the wave is travelling horizontally then the vibration will be up and down, so it is travelling or progressive wave.

2.3 Types of Travelling Wave Solutions

A travelling wave solution describes a stable waveform that moves at a constant velocity without changing shape. It is obtained by reducing a nonlinear evolution equation (NLEE) to an ordinary differential equation (ODE) using the function $w(x,t) = w(\lambda), \lambda = x - vt$, v is the wave velocity. This transformation simplifies the analysis and enables exact solutions through various methods. Travelling waves, such as solitons and kinks, play vital roles in modeling phenomena like shallow water waves, plasma motion, and optical pulses.

2.3.1 Solitary waves and Solitons

Solitary waves are localized travelling waves travelling with constant speeds and shape, asymptotically zero at large distances. Solitons are special kinds of solitary waves. The soliton solution is spatially localized solution, hence $w(\lambda), w''(\lambda) \to \pm\infty$ and $w'''(\lambda) \to \pm\infty$ as $\lambda \to 0$. Solitons have a remarkable soliton property in that it keeps its identity upon interacting with other solitons (Figure 1).

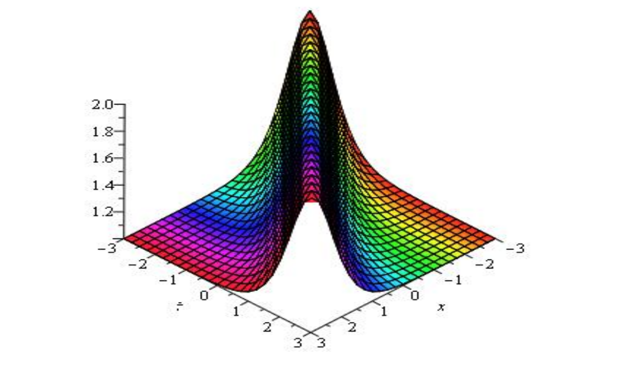

Figure 1: Graph of a soliton solution $w(x,t) = 1 + \sec h^2(x-t); -3 \le x, t \le 3.$

2.3.2 Periodic Solutions

Periodic solutions are travelling wave solutions that are periodic such as $\cos(x-t)$. The standard wave equation $w_{xx} = w_{tt}$ gives periodic solutions. As stated before, because this standard wave equation is linear, it admits D'Alembert solution, and components can be superposed (Figure 2).

2.3.3 Kink waves

Kink waves are travelling waves which rise or descend from one asymptotic state to another. The kink solution approaches a constant at infinity. The standard dissipative Burgers equation $z_t + zz_x = wz_x$, Where w is the viscosity coefficient, is a well-known equation that gives kink solutions (Figure 3).

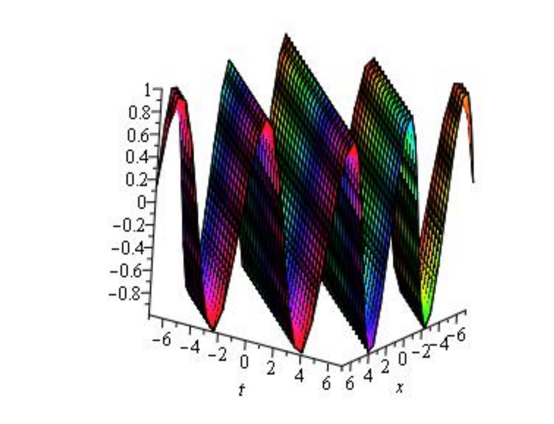

Figure 2: Graph of a periodic solution $w(x,t) = \cos(x-t); -7 \le x, t \le 7$.

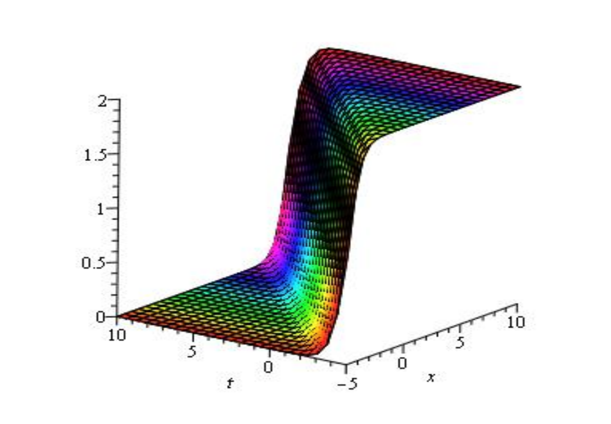

Figure 3: Graph of a kink solution $w(x,t) = 1 + \tanh(x-t); -5 \le x, t \le 15$.

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2.3.4Peakons

Peakons are peaked solitary wave solutions. In this case, the travelling wave solutions are smooth except for a peak at a corner of its crest. Peakons are the points at which spatial derivative changes sign so that peakons have a finite jump in first derivative of the solution $w(x,t)$. This means that peakons have discontinuities in the *x*-derivative but both one-sided derivatives exist and differ only by a sign. The peakons are solitons retaining their shape and speed after interacting (Figure 4).

2.3.5Cuspons

Cuspons are other forms of solitons where solution exhibits cusps at their crests. Unlike peakons where the derivatives at the peak differ only by a sign, the derivatives at the jump of a cuspons diverges (Figure 5).

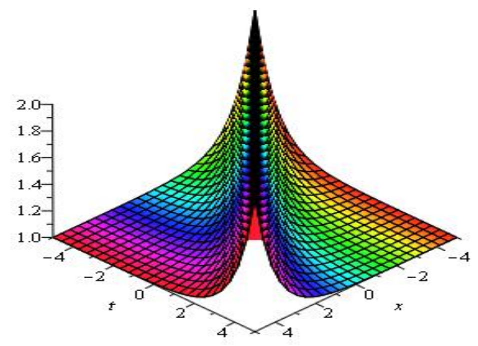

Figure 4. Graph of a peakon solution

$$w(x,t) = 1 + e^{|x-t|}; -5 \le x,t \le 5.$$

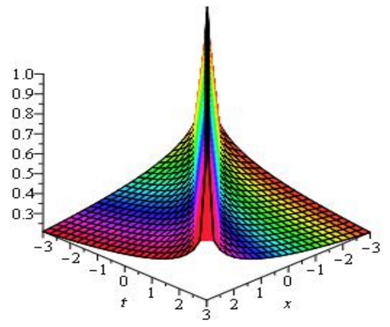

Figure 5. Graph of a cuspon solution

$$w(x,t) = e^{|x-t|^{\frac{1}{5}}}; -3 \le x,t \le 3.$$

2.3.6 Compacton

Compacton is defined by solitary waves with the remarkable soliton property that after colliding with other compactons, they reemerge with the similar coherent shape. These particles like wave's shows elastic collision that are similar to the soliton collision. The following below figure shows that a graph of compacton (Figure 6).

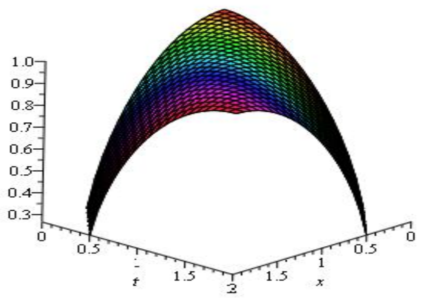

Figure 6. Graph of compacton of

$$w(x,t) = \cos^{\frac{1}{2}}(x-t);\ 0 \le x,t \le 2.$$

3. The Generalized Kudryashov Method

In this part, we describe the main steps of the generalized Kudryashov method for finding exact travelling wave solution of NLEEs. To begin, consider the well-known Riccati equation: $\frac{dR(\psi)}{d\psi} = m + R^2(\psi)$, Where $R = R(\psi)$. The solutions of the Equation are as follows:

Case 1: When $m < 0$, the general solutions are

$$R_1 = -\sqrt{-m}\ \tanh(\sqrt{-m}\psi), \quad (1)$$

$$R_2 = -\sqrt{-m}\coth(\sqrt{-m}\psi). \quad (2)$$

Case 2: When $m > 0$, the general solutions are

$$R_3 = \sqrt{m}\tan(\sqrt{m}\psi), \quad (3)$$

$$R_4 = -\sqrt{m}\cot(\sqrt{m}\psi). \quad (4)$$

Case 3: When $m = 0$, then the general solution is

$$R_5 = -\frac{1}{\psi}, \quad (5)$$

Where, m is a real parameter.

Suppose that we have NLEEs of the form

$$T(w, w_t, w_x, w_{xx}, ..) = 0, x \in \lambda, t > 0, \quad (6)$$

Where $w = w(x,t)$ is an unknown function and T is a polynomial in w and its various partial derivatives, in which the highest-order derivatives and nonlinear terms are involved.

The main steps of generalized Kudryashov method are as follows:

First step: The traveling wave transformation $w(x,t) = w(\psi), \psi = k(x - \omega t)$ transforms Equation (6) into an ordinary differential equation of the form

$$\gamma(w, w', w'', ...) = 0, \quad (7)$$

Where the prime indicates differentiation with respect to ψ.

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Second step: Suppose that the solution of Equation (7) has the following form:

$$w(\psi) = \frac{\sum_{i=0}^{N} a_i R^i(\psi)}{\sum_{j=0}^{M} b_j R^j(\psi)}, \qquad (8)$$

Where $a_i (i = 0,1,2,...,N)$ and $b_j (j = 0,1,2,...,M)$ are constants to be calculated later such that $a_N \neq 0$ and $b_M \neq 0$ and $R = R(\psi)$ satisfies the well-known Riccati equation.

Third step: Determine the positive integer numbers *N* and *M* in Equation (7) by using the homogeneous balance method between the highest-order derivatives and the nonlinear terms of Equation (6).

Final step: Substituting Equations (1) and (8) into Equation (7), we obtain a polynomial in $R^{i-j} (i, j = 0,1,2...) = 0.$ In this polynomial equating the coefficients of all terms of the same powers of R to zero, we obtain a system of algebraic equations that can be solve by using MAPLE and MATHEMATICA to obtain the unknown parameters $a_i (i = 0,1,2,...,N)$ and $b_j (j = 0,1,2,...,M), \omega.$ Consequently, we obtain the exact solutions of Equation (6).

3.1 Applications

In this section, we will apply the generalized Kudryashov method to construct the exact traveling wave solutions transmutable to the solitary wave solutions for the following two NLEEs.

3.1.1 The Kortweg De-Vries Equation

Traveling wave transformation equation $\psi = g(x - \omega t)$ transforms the KdV Equation

$$w_t + \beta w w_x + w_{xxx} = 0. \qquad (9)$$

Where $\frac{\partial \psi}{\partial x} = g.1 = g$ and $\frac{\partial \psi}{\partial t} = g.(-\omega) = -\omega g$.

Then we have, $w_t = \frac{\partial w}{\partial t} = -\omega g w'$ $w_x = g w'$, $w_{xxx} = w''' g^3$.

Into the following ordinary differential equation:

$$-\omega w' + \beta w w' + g^2 w''' = 0. \qquad (10)$$

Integrating Equation (6) with respect to ψ , we get

$$C - \omega w + \beta \frac{w^2}{2} + g^2 w'' = 0. \qquad (11)$$

Now, balancing nonlinear term w^2 and linear term w'' .We get $N = M + 2.$ Setting $M = 1$ yields, $N = 3$.

Hence, Equation (5) gives

$$w(\psi) = \frac{a_0 + a_1 R + a_2 R^2 + a_3 R^3}{b_0 + b_1 R}. \qquad (12)$$

Substituting Equations (9) and (11) into Equation (12), we obtain a polynomial of $R^f, (f = 0,1,2,...).$ Equating the coefficients of all terms of the same power of R to zero. We obtain a system of algebraic equations. This system of equations yields the following values for

$$\omega, a_0, a_1, a_2, a_3, b_0, b_1, H.$$

$$H = -\frac{16g^4m^2 - \omega^2}{2\beta}, a_0 = -\frac{b_0(8mg^2 - \omega)}{\beta}, a_1 = -\frac{b_1(8mg^2 - \omega)}{\beta}, a_2 = -\frac{12b_0g^2}{\beta}, a_3 = -\frac{12b_1g^2}{\beta}.$$

Substituting the values of values for $\omega, a_0, a_1, a_2, a_3, b_0, b_1, H$ and the general solutions of the Riccati equation (1) into Equation (12), we obtain the following traveling wave solutions of the KdV equation (1).

Case 1: When *m* < 0, we obtain the following exact traveling wave solutions with some free parameters in terms of hyperbolic function.

$$w(x,t) = \frac{\omega - 8mg^2}{\beta} + \frac{12mg^2}{\beta} \tanh^2(\sqrt{-m} g(x - \omega t)), \qquad (13)$$

$$and \;\; w(x,t) = \frac{\omega - 8mg^2}{\beta} + \frac{12mg^2}{\beta} \coth^2(\sqrt{-m} g(x - \omega t)). \qquad (14)$$

Case 2: When *m* > 0, we obtain the following exact traveling wave solutions with some free parameters in terms of trigonometric function.

$$w(x,t) = \frac{8mg^2 - \omega}{\beta} - \frac{12mg^2}{\beta} \tan^2(\sqrt{m} g(x - \omega t)), \qquad (15)$$

$$w(x,t) = -\frac{8mg^2 - \omega}{\beta} - \frac{12mg^2}{\beta} \cot^2(\sqrt{m} g(x - \omega t)). \qquad (16)$$

Case 3: When *m*=0, we obtain the following exact traveling wave solution with some free parameters in terms of rational function.

$$w(x,t) = \frac{\omega}{\beta} - \frac{12}{\beta(x - \omega t)^2}.$$

3.1.2 Regularized Long Wave Equation

Traveling wave transformation equation $\psi = g(x - \omega t)$ transforms the regularized long-wave equation:

$$w_t + w_x + \beta w w_x - w_{xxt} = 0 \qquad (18)$$

Where $\frac{\partial \psi}{\partial x} = g.1 = g$ and $\frac{\partial \psi}{\partial t} = g.(-\omega) = -\omega g$.

Then we have,

$$w_t = \frac{\partial w}{\partial t} = -\omega g w', w_x = \frac{\partial w}{\partial x} = g w', w_{xxt} = \frac{\partial^3 w}{\partial x^2 . \partial t} = -\omega g^3 w'''.$$

Into the following ordinary differential equation:

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$$-\omega w' + w' + \beta w w' + g^2 \omega w''' = 0. \quad (19)$$

Integrating Equation (1) with respect to w, we have

$$D - (\omega - 1)w + \frac{\beta}{2} w^2 + g^2 \omega w'' = 0. \quad (20)$$

According to step 3, $N = M + 2$. Hence, $M = 1$ yields $N = 3$.

Therefore, equation (14) gives

$$w(\psi) = \frac{a_0 + a_1 R + a_2 R^2 + a_3 R^3}{b_0 + b_1 R}, \quad (21)$$

Substituting Equation (21) into Equation (20), we obtain a polynomial of, R^f, $(f = 0,1,2,\ldots)$ Equating the coefficients of all terms of the same power of *A* to zero, we attain a system of algebraic equation. This system of equations yields the following values for $\omega, a_0, a_1, a_2, a_3, b_0, b_1, D$.

$$D = -\frac{(16g^4 m^2 - 1)\omega^2 + 2\omega - 1}{2\beta},$$

$$a_0 = -\frac{b_0(8m\omega g^2 - \omega + 1)}{\beta},$$

$$a_1 = -\frac{b_1(8m\omega g^2 - \omega + 1)}{\beta}, a_2 = -\frac{12 b_0 \omega g^2}{\beta},$$

$$a_3 = -\frac{12 b_1 \omega g^2}{\beta}$$

Substituting the values of values for $\omega, a_0, a_1, a_2, a_3, b_0, b_1, D$ and the general solutions of the Riccati equation (1) into Equation (16), we obtain the following traveling wave solutions of the RLW equation (18).

Case 1: When $m < 0$, we obtain the following exact traveling wave solutions with some free parameters in terms of hyperbolic function.

$$w(x,t) = \frac{\omega(1 - 8mg^2) - 1}{\beta} + \frac{12\omega m g^2}{\beta} \tanh^2(\sqrt{-m}g(x - \omega t)), \quad (22)$$

$$w(x,t) = \frac{\omega(1 - 8mg^2) - 1}{\beta} + \frac{12\omega m g^2}{\beta} \coth^2(\sqrt{-m}g(x - \omega t)). \quad (23)$$

Case 2: When $m > 0$, we obtain the following exact traveling wave solutions with some free parameters in terms of trigonometric function.

$$w(x,t) = \frac{\omega(8mg^2 - 1) + 1}{\beta} - \frac{12\omega m g^2}{\beta} \tan^2(\sqrt{m}g(x - \omega t)), \quad (24)$$

$$w(x,t) = \frac{\omega(8mg^2 - 1) + 1}{\beta} - \frac{12\omega m g^2}{\beta} \cot^2(\sqrt{m}g(x - \omega t)). \quad (25)$$

Case 3: When $m = 0$, we obtain the following exact traveling wave solution with some free parameters in terms of rational function.

$$w(x,t) = \frac{\omega - 1}{\beta} - \frac{12\omega}{\beta(x - \omega t)^2}. \quad (26)$$

Graphical Representation

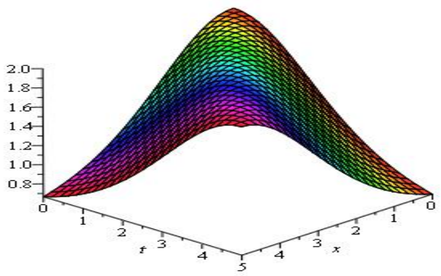

Figure 1(a). Dark soliton shape of KdV equation $\omega = g = \beta = 1$ and $m = -0.150$ (only shows the shape of Equation (13)).

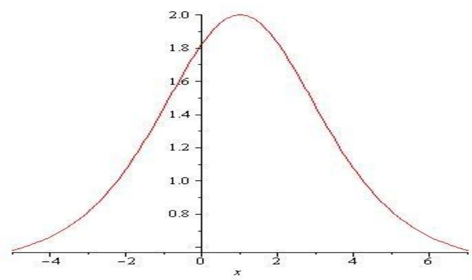

Figure 1(b). Corresponding 2D graph of figure 1for $\omega = g = \beta = 1$ and $m = -0.150$, $t = 1$ (only shows the shape of Equation (13)).

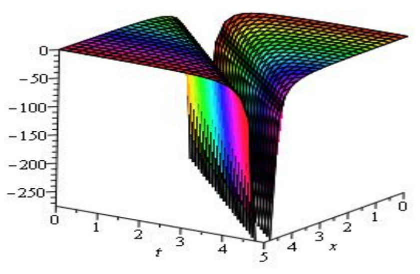

Figure 2(a). Dark singular soliton solution of KdV equation for $\omega = \beta = k = 1$ and $m = -0.125$ (only shows the shape of Equation (14)).

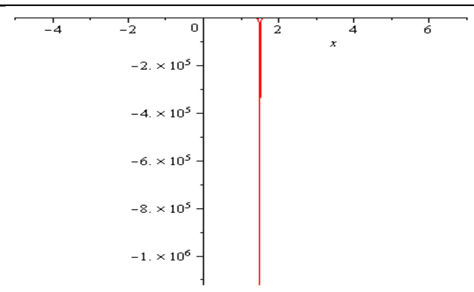

Figure 2(b). Corresponding 2D graph of figure 2 for $\omega = \beta = k = 1$ and $m = -0.125, t = 1.5$ (only shows the shape of Equation (14).

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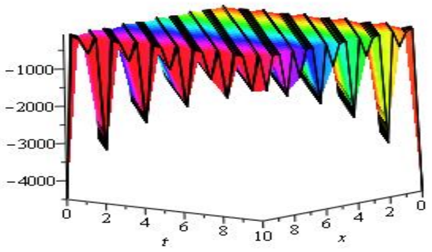

Figure 3(a). Periodic shape of KdV equation for $m=\omega=\beta=1$ and $k=5$ (only the shape of Equation (15)).

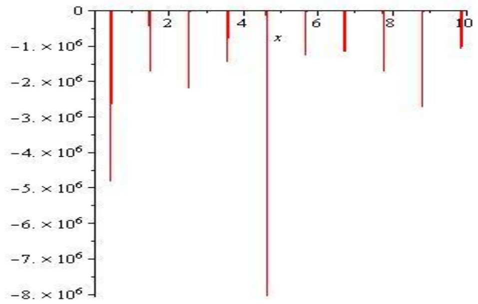

Figure 3(b). Corresponding 2D graph of figure 3 for $m=\omega=\beta=1$ and $k=5, t=1$ (shape of Equation (15)).

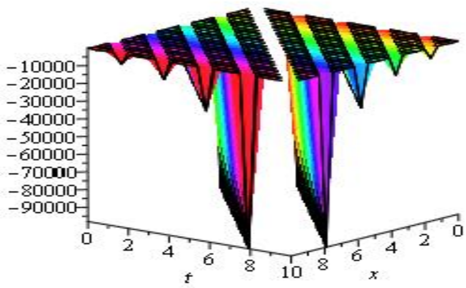

Figure 4(a). Periodic shape of KdV equation for $m=\omega=\beta=1$ and $k=4$ (only shows the shape of Equation (16)).

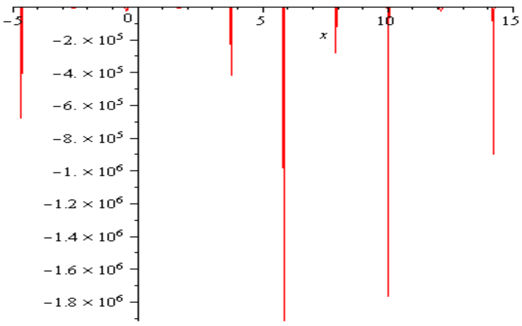

Figure 4(b). Corresponding 2D graph of figure 4 for $m=\omega=\beta=1$ and $k=4, t=-1.5$ (only shows the shape of Equation (16)

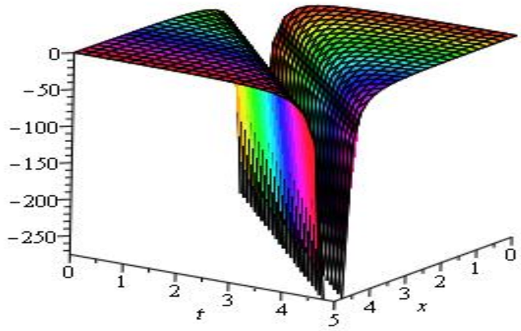

Figure 5(a). Dark singular soliton solution of KdV equation for $\omega=\beta=k=1$ and $m=-0.150$ (only shows the shape of Equation (17))

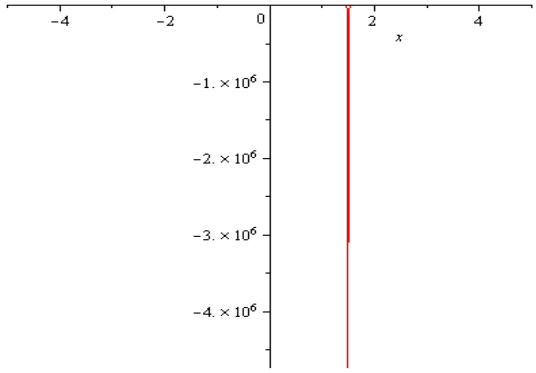

Figure 5(b). Corresponding 2D graph of figure 5 for $\omega=\beta=1$ and $t=1.5$ (only shows the shape of Equation (17)).

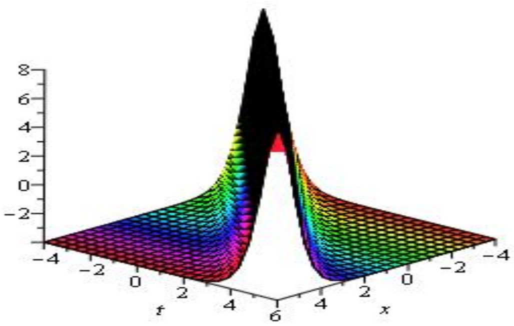

Figure 6(a). Dark soliton shape of regularized long-wave equation for $\omega=g=\beta=1$ and $m=-3$ (only shows the shape of Equation (22)).

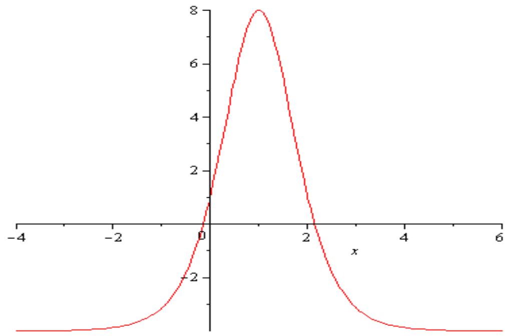

Figure 6(b). Corresponding 2D graph of figure 6 for $\omega=g=\beta=1, m=-3, t=1$ (only shows the shape of Equation (22)).

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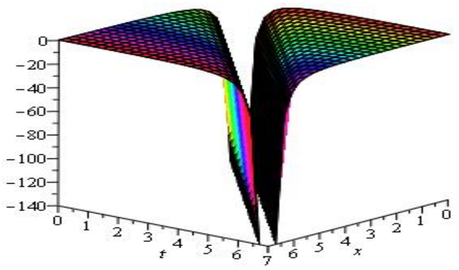

Figure 7(a). Dark singular soliton solution of regularized long wave equation for $\omega = \beta = k = 1$ and $m = -1$ (only shows the shape of Equation (23)).

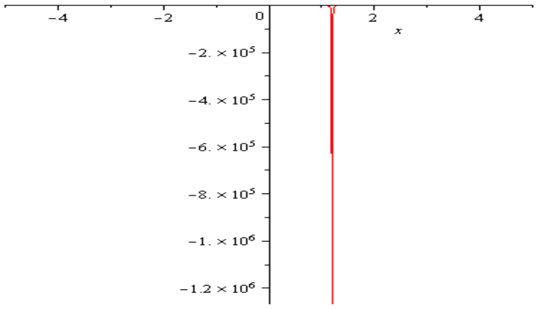

Figure 7(b). Corresponding 2D graph of figure 7 for $\omega = \beta = k = 1$ and $m = -1, t = 1.2$ (only shows the shape of Equation (23)).

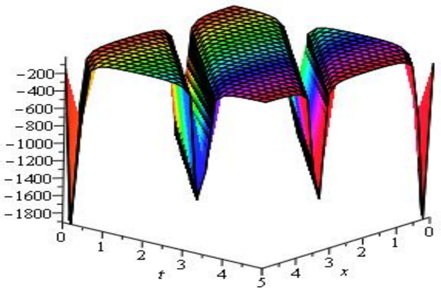

Figure 8(a). Periodic shape of regularized long-wave equation for $\omega = m = \beta = g = 1$ (only shows the shape of Equation (24)).

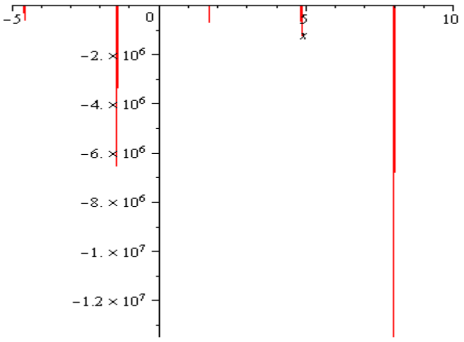

Figure 8(b). Corresponding 2D graph of figure 8 for $\omega = m = \beta = g = 1$ and $t = -3$ (shape of Equation (24)).

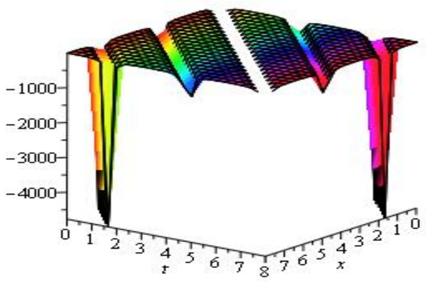

Figure 9(a). Periodic shape of regularized long-wave equation for $m = \omega = \beta = 1$ and $k = 4$ (only shows the shape of Equation (25))

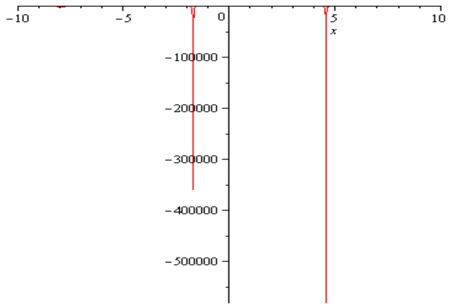

Figure 9(b). Corresponding 2D graph of figure 9 for $m = \omega = \beta = 1$ and $k = 4, t = -1.7$ (only shows the shape of Equation (25)).

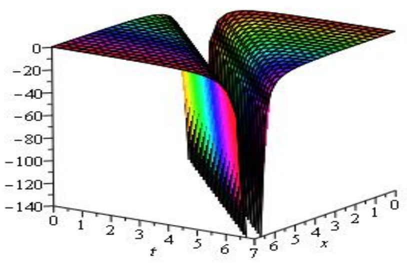

Figure 10(a). Dark singular soliton solution of regularized long wave equation for $\omega = \beta = k = 1$ and (Eq. 26)).

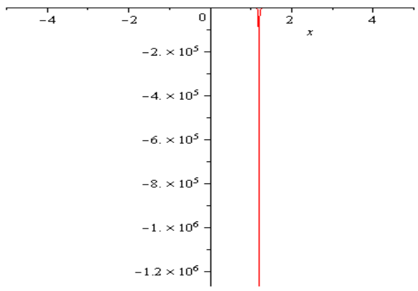

Figure 10(b). Corresponding 2D graph of figure 10 for $\omega = \beta = k = 1$, $m = -1.25, t = 1.2$ (Equation (26)).

4. The MSE Method

The Modified Simple Equation (MSE) method expresses exact solutions of nonlinear equations as polynomial forms involving an auxiliary function or its derivative. this method [34, 35, 36, 37] is based on the assumptions that the exact solutions can be expressed by a polynomial in ψ, such that $\psi = \psi(w)$ satisfy in the equations of Bernoulli and Riccati which are well known nonlinear ordinary differential equations and their solutions can be expressed by elementary functions. Through this formulation, the MSE

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method provides a reliable and straightforward technique for solving nonlinear ordinary differential equations (ODEs). It has been effectively applied to models like the Sharma–Tasso–Olver, generalized Korteweg–de Vries (KdV), and reaction–diffusion equations. In the present study, the method is utilized to determine exact analytical solutions of a second-order nonlinear ODE.

In this section we describe the MSE method for finding travelling wave solution of the nonlinear evolution equations. Let us suppose the nonlinear equation of two independent variables x and t is given by $\lambda(w, w_t, w_x, w_{xt}, w_{xx}, \ldots) = o$, (27)

Where $w(x,t)$ is an unknown function, λ is a polynomial of $w(x,t)$ and its partial derivatives in which the highest order derivative and the nonlinear terms are involved here. In the following, we give the main steps of the modified simple equation method:

In the starting combine the two independent variables x and t into a compound variable

$\psi = \mathrm{x} - \omega \mathrm{t}$, let us assume that, $w(\psi) = w(x,t)$ where

$$\psi = \mathrm{x} - \omega \mathrm{t} \quad (28)$$

The travelling wave transformation Eq. (28) permits us to reduce the Eq. (27) into the given ordinary differential equation (ODE):

$$\lambda(w, w', w'', w''', \ldots) = 0, \quad (29)$$

where λ is a polynomial of $w(\psi)$ and its derivative, while $w'(\psi) = \dfrac{du}{d\psi}, w''(\psi) = \dfrac{d^2u}{d\psi^2}$, and so on.

Step 1: We assume that the solution of Eq. (29) can be presented in the following form

$$w(\psi) = D_0 + \sum_{k=1}^{n} D_k \left(\frac{\phi'(\psi)}{\phi(\psi)}\right)^k \quad (30)$$

Where $C_k (k = 1,2,\ldots)$ are the arbitrary constant to be determined, such that $C_n \neq 0,$ and $\phi(\psi)$ is an unknown function to be determined later.

Step 2: In this step we determined the positive integer n come out in Eq. (30) by considering the homogeneous balance between the highest order derivatives and the non-linear terms in the Eq. (29).

Step 3: we substitute Eq. (30) into (29) and then we account the function $\phi(\psi)$. As a result of this substitution, we will get a polynomial of $\dfrac{\phi'(\psi)}{\phi(\psi)}$ and its derivatives. In this polynomial, we equate the coefficients of the same power of $\phi^{-i}(\psi)$ to zero, where $i \geq 0$. This procedure yields a system of equations which can be solved to find C_k, $\phi(\psi)$ and $\phi'(\psi)$. Then the substitution of the values of C_k, $\phi(\psi)$ and $\phi'(\psi)$ into in Eq. (30) complete the determination of exact solution of Eq.(27) .

4.1 Applications

The coupled (1+1) dimensional Broer–Kaup (BK) equation

The BK equation describes the bi-directional propagation of long wave in shallow water. Now we will bring to bear the MSE method to find the exact solutions and then the solitary wave solutions to the BK equation in the form,

$$w_t = ww_x + z_x - \frac{1}{2} w_{xx} \ \ and \ \ z_t = (wz)_x + \frac{1}{2} z_{xx}. \quad (31)$$

Suppose that the traveling wave transformation equation be

$w(\psi) = w(x,t), z(\psi) = z(x,t)$, where $\psi = x - \omega t$ (32)

The wave transformation (32) reduces Eq. in (31) into the following ODEs

$$-\omega w' - ww' - z' + \frac{1}{2} w'' = 0 \quad (33)$$

$$\omega z' + (wz)' + \frac{1}{2} z'' = 0 \quad (34)$$

Now, integrating Eq. (33) and (34) with respect to ψ, and neglecting the constant of integration we obtain,

$$-\omega w - \frac{1}{2} w^2 - v + \frac{1}{2} w' = 0. \quad (35)$$

$$\omega z + (wz) + \frac{1}{2} z' = 0. \quad (36)$$

Eq. (35) becomes $z = -\omega w - \dfrac{1}{2} w^2 + \dfrac{1}{2} w'.$ (37)

Substituting Eq. (37) into Eq. (36), we obtain

$$\omega^2 w + \frac{3}{2} \omega w^2 + \frac{1}{2} w^3 - \frac{1}{4} w'' = 0. \quad (38)$$

Balancing the highest order derivative w'' and nonlinear term w^3 from Eq. (38), we obtain 3n = n+2, which gives n=1.

Therefore, Eq. (30) can be written as $w(\psi) = D_0 + D_1 \left(\dfrac{\phi'(\psi)}{\phi(\psi)}\right).$ Where D_0 and D_1 are constants to be determined such that, while is an unknown function to be determined. It is problem-free to find that,

$$w' = D_1 \left\{\frac{\phi''}{\phi} - \left(\frac{\phi'}{\phi}\right)^2\right\}. \quad (39)$$

$$w'' = D_1 \left(\frac{\phi'''}{\phi}\right) - 3D_1 \left(\frac{\phi''\phi'}{\phi^2}\right) + 2D_1 \left(\frac{\phi'}{\phi}\right)^3. \quad (40)$$

$$w^2 = D_0^{\ 2} + 2D_0 D_1 \left(\frac{\phi'}{\phi}\right) + D_1^{\ 2} \left(\frac{\phi'}{\phi}\right)^2. \quad (41)$$

$$w^3 = D_0^{\ 3} + 3D_0^{\ 2} D_1 \left(\frac{\phi'}{\phi}\right) + 3D_0 D_1^{\ 2} \left(\frac{\phi'}{\phi}\right)^2 + D_1^{\ 3} \left(\frac{\phi'}{\phi}\right)^3. \quad (42)$$

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Now substituting the values of w, w^2, w^3, w'' into Eq. (37) and then equating the coefficients of $\phi^0, \phi^{-1}, \phi^{-2}, \phi^{-3}$ to zero, we respectively obtain,

$$\frac{1}{2}D_0^{\ 3} + \frac{3}{2}\omega D_0^{\ 2} + \omega^2 D_0 = 0. \qquad (43)$$

$$\omega^2 D_1\phi' - \frac{1}{4}D_1\phi''' + \frac{3}{2}D_0^{\ 2}D_1\phi' + 3\omega D_0 D_1\phi' = 0. \qquad (44)$$

$$\frac{3}{2}\omega D_1^{\ 2}(\phi')^2 + \frac{3}{2}D_0 D_1^{\ 2}(\phi')^2 + \frac{3}{4}D_1\phi'\phi'' = 0. \qquad (45)$$

$$\frac{1}{2}D_1^{\ 3}(\phi')^3 - \frac{1}{2}D_1(\phi')^3 = 0. \qquad (46)$$

Solving Eq. (43), we obtain $D_0 = 0, -\omega, -2\omega$.

Solving Eq. (46), we obtain $D_1 = \pm 1$, since $D_1 \neq 0$.

Solving Eq. (44) and (45), we obtain $\phi'(\psi) = GH\exp(GM\psi)$. (47)

Integrating, Eq. (46) with respect to ψ, we get

$$\phi(\psi) = \frac{1}{M}(H\exp(GM\psi) + MR). \qquad (48)$$

Where $M = 4\omega^2 + 6D_0^{\ 2} + 12D_0, G = -\dfrac{1}{2D_1(\omega + D_0)}$

and *A*, *B* are constants of integration.

Substituting the values of and into Eq. (39), we obtain the following exact solution i.e.

$$w(x,t) = D_0 + D_1\frac{MGH\exp(GM(x-\omega t))}{H\exp(GM(x-\omega t)) + MR}. \qquad (49)$$

Case-I: When $D_0 = 0, -2\omega$ and $D_1 = -1$, also when $D_0 = -\omega$ and $D_1 = \pm 1$, Eq. (49) yields trivial solutions. So this case is discarded.

Case-II: When $D_0 = 0$ and $D_1 = 1$, putting the values of into Eq. (49) and then simplifying, so we will get,

$$w(x,t) = -2\omega H\{\frac{\cosh(\omega(x-\omega t)) + \sinh(\omega(x-\omega t))}{(H+4\omega^2 R)\cosh(\omega(x-\omega t)) - (H-4\omega^2 R)\sinh(\omega(x-\omega t))}\} \qquad (50)$$

We can freely choose the constants H and R. Therefore, substituting the value of G, M and then setting $H = 4\omega^2$ and $R = 1$, Eq. (50) reduces to

$$w_1(x,t) = \omega(\tanh(\omega(x-\omega t)) - 1) \qquad (51)$$

Substituting Eq. (51) into Eq. (38), we obtain

$$z_1(x,t) = \omega^2 \sec h^2(\omega(x-\omega t)). \qquad (52)$$

Again, if we set and, Eq. (50) reduces to

$$w_2(x,t) = \omega(\coth(\omega(x-\omega t)) - 1) \qquad (53)$$

Substituting Eq. (53) into Eq. (38), we obtain

$$z_2(x,t) = -\omega^2 \cos ech^2(\omega(x-\omega t)). \qquad (54)$$

Case- III: When $D_0 = -2\omega$ and $D_1 = 1$, we obtain same results included in Eq. (51)-(54).

4.1.1Graphical Representation

In this section, we will discuss about the nature and the shape of figure of the above equations 51-54 of the above ***Case-II.***

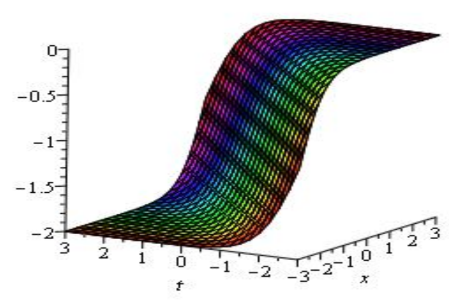

Figure 6(a): Shape of the equation (51) for $\omega = 1$ and in the interval $-3 \leq x, t \leq 3$.

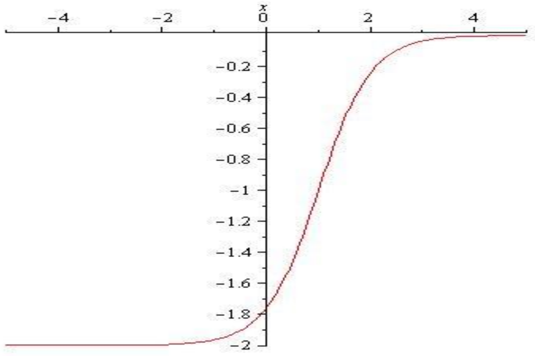

Figure 6(b): Corresponding 2D graph of figure 1 for $\omega = 1$ and $-3 \leq x, t = 1$.

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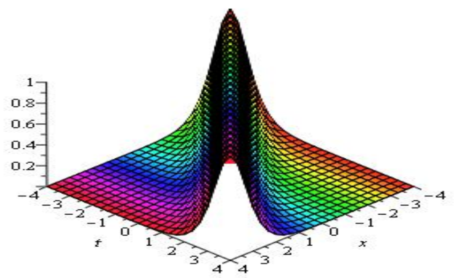

Figure 7(a): The shape of equation (52) for $\omega = 1$ and $-4 \le x, t \le 4.$

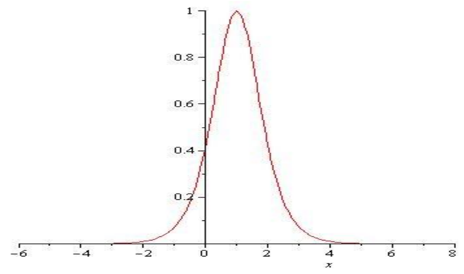

Figure 7(b): Corresponding 2D graph of figure 2 for $\omega = 1$ and $-3 \le x, t = 1.5.$

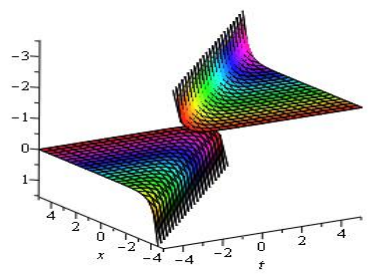

Figure 8(a): The shape of equation (53) for $\omega = 1$ and $-5 \le x, t \le 5.$

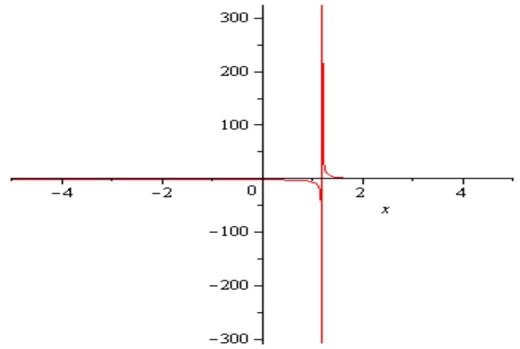

Figure 8(b): Corresponding 2D graph of figure 3 for $\omega = 1$ and $-3 \le x, t = 3.$

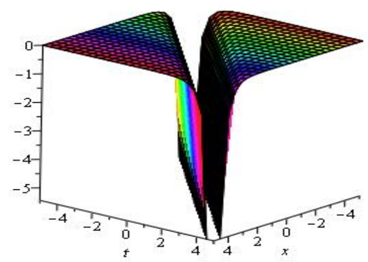

Figure 9(a): The shape of equation (54) for $\omega = 1$ and $-5 \le x, t \le 5.$

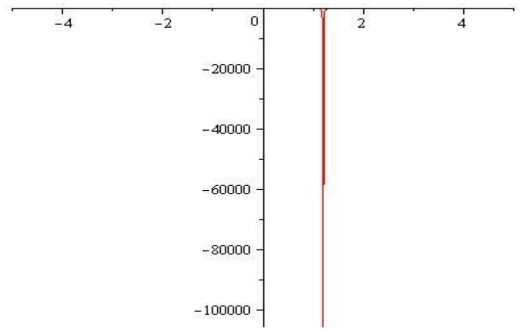

Figure 9(b): Corresponding 2D graph of figure 4 for $\omega = 1$ and $-5 \le x, t = 1.4.$

5. Results and Discussion

We investigated the travelling wave solutions of KdV equations and its alternatives by using Generalized Kudryashov methods. Three basic types of wave solutions, such as dark solitons shape, singular soliton solution shape and periodic soliton solutions shape were found. For a specific value of the parameter ($\omega = 1$) the solutions of the RLW equation coincide with those of the KdV equation. The obtained KdV solutions exhibit different wave patterns depending on parameter values: when certain conditions are met, solitary waves appear, while other parameter choices produce plane periodic travelling waves. By selecting suitable parameter values, solitary wave structures can be derived from the general travelling wave solutions. Several figures are presented to illustrate these solitary waves and to visualize the fundamental behavior of the original equation. Finally, for solving the coupled (1+1) dimensional Broer–Kaup (BK) equation by using MSE method we have generated from equation (51) represents the kink soliton, which transitions smoothly between two asymptotic states and is known as a topological solution. Figure 6 illustrates this behavior, showing that the wave approaches a constant value at infinity. Equation (52) describes a bell-shaped soliton, a non-topological solution with multiple oscillatory wings, as shown in Figure 7 (for $\omega = 1$ and the interval $-4 \le x, t \le 4$). Equations (53) and (54) yield singular soliton solutions, special solitary waves lacking a defined midpoint, often associated with rogue wave phenomena, as depicted in Figures 8 and 9 (for $\omega = 1$ and the interval $-5 \le x, t \le 5$). Moreover, the results indicate that for positive wave velocity, the disturbance moves in the positive x-direction, while for negative velocity, it propagates in the opposite direction.

6. Conclusion

In this research work, we have mainly focused on the new exact travelling wave solutions of nonlinear evolution equations

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(NLEEs) are obtained using the Modified Simple Equation (MSE) method and the Generalized Kudryashov Method (GKM) with the help of mathematical software such as MAPLE. The MSE method is applied to the coupled (1+1)-dimensional Broer–Kaup (BK) equation, while the GKM is used to derive solutions for the omnipotent unsteady Korteweg–de Vries (KdV) equation and its variant, the Regularized Long Wave (RLW) equation. The research presents exact solutions expressed in three functional forms—hyperbolic, trigonometric, and rational and includes both 2D and 3D graphical representations of selected results. The findings demonstrate that these methods are powerful and reliable analytical tools for determining exact solutions of various NLEEs in nonlinear science, efficiently implemented using MAPLE.

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