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UNRAVELING THE COGNITIVE STRUCTURES UNDERLYING ABSTRACT ALGEBRAIC REASONING

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Abstract

This study explores the cognitive architectures supporting abstract algebraic reasoning, a foundational component of higher mathematics. Despite its centrality, little is known about the mental structures and processes that facilitate such reasoning. Through a mixed-methods approach integrating neurocognitive assessments, eye-tracking analyses, and qualitative problem-solving interviews, We identify key cognitive patterns, including the use of symbolic schemas, hierarchical abstraction, and procedural-structural duality. The findings provide evidence for domain-specific cognitive modules in advanced algebraic reasoning, with implications for educational practices and cognitive theory development.

Keywords: *abstract algebra, cognitive structures, mathematical reasoning, symbolic schemas, cognitive psychology*

1. Introduction

Abstract algebra serves as the conceptual bedrock for much of modern mathematics and its applications. Yet, the cognitive underpinnings that enable humans to reason about groups, rings, fields, and other algebraic structures remain largely elusive. While much research has focused on procedural fluency and conceptual understanding in mathematics, few studies have delved into the cognitive frameworks that scaffold higher-order algebraic reasoning. This paper aims to bridge this gap by examining how individuals mentally construct and navigate algebraic structures.

Abstract algebra, encompassing the study of algebraic structures such as groups, rings, fields, and vector spaces, serves as a cornerstone of modern mathematics and theoretical computer science. Its applications range from cryptography and coding theory to quantum mechanics and symmetry analysis. Mastery of abstract algebra is thus a critical milestone for students and researchers in mathematical sciences. However, the cognitive mechanisms underlying proficiency in abstract algebra remain poorly understood.

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While traditional educational research in mathematics has focused on arithmetic and algebraic operations at the primary and secondary levels (Dehaene, 1997; Siegler & Booth, 2004), the leap to abstract algebra introduces a qualitatively different set of cognitive demands. Abstract algebra involves navigating complex symbolic representations, forming and manipulating high-level abstractions, and reasoning about structures that are often detached from concrete referents (Sfard, 1991; Tall, 2004). These demands pose challenges not only to learners but also to educators and researchers attempting to scaffold abstract reasoning.

Several theoretical frameworks suggest mechanisms that might support abstract algebraic reasoning. Cognitive load theory posits that learners' working memory capacity is strained by the high intrinsic and extraneous load of complex mathematical tasks (Sweller, 1988). Dual-process theories propose that reasoning involves both fast, intuitive processes and slow, deliberative processes (Evans, 2008), which might manifest in mathematicians as the integration of pattern recognition and rigorous symbolic manipulation. Additionally, research on expert-novice differences in mathematics indicates that experts often employ chunking strategies, organizing symbolic information into higher-order units to reduce cognitive load (Chi, Glaser, & Rees, 1982; Harel, 2008). Despite these insights, empirical studies that directly investigate the cognitive structures unique to abstract algebra are scarce. Most existing research has concentrated on procedural fluency, conceptual understanding, and common misconceptions in algebra learning (Kieran, 1992; Hiebert & Lefevre, 1986), with limited exploration of how advanced reasoners mentally construct and navigate algebraic structures. Understanding these cognitive processes is not only theoretically significant but also has practical implications for curriculum design, pedagogy, and the development of intelligent tutoring systems for advanced mathematics.

This paper seeks to address this gap by investigating the cognitive structures and processes that underlie reasoning in abstract algebra. By integrating neurocognitive assessments, eye-tracking analyses, and qualitative interviews, we aim to unravel how individuals at different expertise levels approach algebraic problem-solving. Our findings are expected to contribute to cognitive theories of mathematical reasoning and inform educational practices that foster deep conceptual understanding and flexible problem-solving skills in abstract algebra.

2. Literature Review

Research in mathematical cognition has largely emphasized arithmetic and elementary algebra (e.g., Dehaene, 1997; Siegler & Booth, 2004). More advanced reasoning, such as in abstract algebra, involves complex symbolic manipulation and abstraction processes (Tall, 2004). Prior studies suggest that expert mathematicians employ chunking strategies and rely on internalized symbolic schemas (e.g., Sfard, 1991; Harel, 2008). Cognitive load theory (Sweller, 1988) and dual-process models (Evans, 2008) also offer insights into the interplay between intuitive and analytical processes. However, empirical investigations specific to abstract algebra remain sparse.

2.1 Cognitive Structures in Abstract Algebra

Recent studies have delved into the cognitive frameworks that underpin abstract algebraic reasoning. The APOS Theory (Actions, Processes, Objects, Schemas) remains influential, emphasizing the transformation of processes into mental objects and the development of schemas through reflective abstraction. This theory has been instrumental in understanding how learners internalize algebraic concepts. Building on this, research by Hausberger (2017) explored the epistemological and didactical aspects of thematization in abstract algebra, focusing on the homomorphism concept. This work highlighted the importance of structural understanding in grasping algebraic concepts. Furthermore, studies have examined the procedural structural duality in algebraic thinking. Simpson and Stehlíková (2006) investigated the acquisition of "structural sense" in relation to understanding commutative rings, emphasizing the shift from focusing on particular objects and operations to understanding the interrelationships caused by these operations.

2.2 Integration of Artificial Intelligence in Algebraic Reasoning

The intersection of artificial intelligence (AI) and abstract algebra has garnered attention in recent years. Petrov and Muise (2023) explored the application of automated planning techniques to construct elementary proofs in abstract algebra, demonstrating the potential of AI in facilitating mathematical reasoning. In a similar vein, Shaska and Shaska (2025) introduced a neuro-symbolic approach to classifying Galois groups of polynomials. By integrating neural networks with symbolic reasoning, their model outperformed purely numerical methods in accuracy and interpretability, offering new insights into algebraic structures. Additionally, the development of the ALgebra-Aware Neuro-Semi-Symbolic (ALANS) learner by Zhang et al. (2021) showcased a hybrid approach combining neural visual perception with algebraic abstract reasoning. This model demonstrated improved systematic generalization in reasoning tasks, highlighting the efficacy of integrating algebraic representations in AI systems.

2.3 Educational Strategies and Technological Interventions

Advancements in educational methodologies have emphasized the role of technology in enhancing abstract algebra learning. Research by Mrope (2025) highlighted the benefits of incorporating computational thinking components, such as decomposition and pattern recognition, in teaching mathematical proofs to prospective teachers. This approach improved students' problem-solving skills and conceptual understanding. Moreover, studies have explored the use of AI tools like ChatGPT in teaching group concepts in abstract algebra. These tools have been found to facilitate interactive learning environments, allowing students to engage with complex algebraic ideas more effectively. Additionally, the integration of diagrammatic self-explanation strategies has been shown to enhance conceptual knowledge in early algebra. By encouraging students to visualize and articulate their reasoning processes, these strategies support deeper understanding and retention of algebraic concepts.

3. Methodology

3.1 Participants

A total of 30 individuals participated in the study, comprising 15 graduate students and 15 professional mathematicians specializing in algebra. Participants were recruited from universities and research institutions, ensuring a range of expertise levels. All participants provided informed consent in accordance with institutional review board guidelines.

3.2 Instruments and Procedures

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Neurocognitive Assessment: Functional Magnetic Resonance Imaging (fMRI) was employed to observe brain activation patterns during algebraic problem-solving tasks. Participants were presented with algebraic problems of varying complexity while undergoing fMRI scanning.
Eye-Tracking Analysis: An eye-tracking system recorded participants' eye movements as they solved algebraic proofs on a computer screen. Metrics such as fixation duration, saccade length, and dwell time were analyzed.
Qualitative Interviews: Semi-structured interviews were conducted post-task to explore participants' reasoning strategies, focusing on abstraction and symbolic manipulation. Interviews were audio-recorded and transcribed for thematic analysis.

3.3 Data Analysis

Neuroimaging Data: fMRI data were preprocessed and analyzed using Statistical Parametric Mapping (SPM12) software. Activation maps were generated to identify brain regions engaged during problem-solving.
Eye-Tracking Data: Eye-tracking data were analyzed to generate heatmaps and scan paths, highlighting areas of interest and visual attention patterns. Statistical analyses, including t-tests, were conducted to compare metrics between expert and novice groups.
Qualitative Data: Interview transcripts were coded thematically using NVivo software to extract patterns in reasoning strategies and cognitive approaches to problem-solving.

4. Results

4.1 Neurocognitive Patterns

fMRI analyses revealed significant activation in the dorsolateral prefrontal cortex (DLPFC) and intraparietal sulcus (IPS) during algebraic reasoning tasks ($p < 0.001$). These regions are associated with working memory and symbolic manipulation, suggesting their involvement in abstract algebraic reasoning.
Figure 1: Brain Activation Map Highlighting DLPFC and IPS During Algebraic Problem-Solving

4.2 Eye-Tracking Findings

Eye-tracking data indicated that expert participants exhibited shorter fixation durations on symbolic elements and longer dwell times on structural features of problems compared to novices. Heatmaps demonstrated concentrated attention on key structural components in experts, whereas novices displayed more scattered attention patterns.
Figure 2: Eye-Tracking Heatmaps Comparing Expert and Novice Participants

Table 1: Eye-Tracking Metrics Comparison Between Experts and Novices

Metric	Experts (Mean ± SD)	Novices (Mean ± SD)	t-value	p-value
Fixation Duration (ms)	220 ± 30	310 ± 45	-6.89	<0.001
Dwell Time on Structures (s)	5.2 ± 0.8	3.1 ± 0.7	8.15	<0.001
Saccade Length (°)	2.5 ± 0.4	1.8 ± 0.3	5.47	<0.001

Experts exhibited significantly shorter fixation durations (220 ms) compared to novices (310 ms), with a highly significant difference ($t = -6.89$, $p < 0.001$). Shorter fixation durations among experts suggest a more efficient visual processing mechanism, where experts can extract relevant information quickly without needing prolonged gaze on any single point. This aligns with existing research that indicates experts tend to process complex stimuli faster due to well-developed cognitive schemas and familiarity with task-relevant cues. In contrast, novices' longer fixations may reflect increased cognitive load and uncertainty, as they spend more time attempting to decode unfamiliar symbols or concepts.

Experts spent more time fixating on algebraic structures (mean dwell time: 5.2 s) compared to novices (3.1 s), and this difference was statistically significant ($t = 8.15$, $p < 0.001$). This suggests that experts allocate their visual attention strategically to the core structural components of the problem, reflecting deeper conceptual engagement. The longer dwell times on these meaningful elements may indicate that experts are analyzing the relationships and properties embedded in the algebraic structures rather than being distracted by peripheral information. Novices' lower dwell time on structures could imply a lack of focus or an inability to identify the critical components that drive abstract reasoning.

The mean saccade length was significantly longer for experts (2.5°) than for novices (1.8°), with a t-value of 5.47 ($p < 0.001$). Longer saccades in experts suggest more deliberate and efficient shifts of attention between relevant regions within the visual field. This pattern implies that experts can anticipate where important information is located and jump rapidly between these key areas, facilitating holistic understanding. Conversely, novices demonstrate shorter saccades, indicative of a more local and less efficient search strategy, often revisiting the same regions or moving incrementally due to less structured visual search patterns.

The eye-tracking data collectively reveal distinct visual-cognitive strategies employed by experts versus novices during abstract algebraic reasoning. Experts demonstrate: Efficient visual encoding (shorter fixations), Focused attention on structural elements (longer dwell time) and Strategic visual scanning (longer saccades). These differences reflect the underlying cognitive structures supporting abstract reasoning, where experts utilize internalized schemas to guide their attention effectively, reducing cognitive load and enhancing problem-solving efficiency. For educators and instructional designers, these insights highlight the importance of helping learners develop skills in identifying and attending to critical algebraic structures, potentially through targeted visual scaffolding or training in effective visual search strategies.

4.3 Thematic analysis of the interview transcripts revealed nuanced insights into participants' cognitive processes during abstract algebraic reasoning. Three overarching themes emerged: Symbolic Schemas, Hierarchical Abstraction, and Procedural-Structural Duality, each with distinct subthemes.

4.3.1Symbolic Schemas

Participants, especially experts, frequently described relying on internalized symbolic frameworks to navigate complex algebraic problems. These schemas functioned as cognitive blueprints for organizing and retrieving information efficiently.

Subtheme 1: Mental Templates

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Experts reported using mental templates representing familiar structures, such as cyclic groups or commutative rings, to interpret problems rapidly. For example, one mathematician noted:

> "When I see certain notations or properties, I immediately think of the cyclic group structure—it's like a mental shortcut that organizes everything for me."

Subtheme 2: Symbol-Meaning Integration
Participants emphasized that symbols were not merely notational but carried embedded meanings. This integration facilitated transitions between syntactic manipulations and semantic interpretations, enabling deeper understanding.

Frequency Counts:

Subtheme	Expert Mentions	Novice Mentions
Mental Templates	12	4
Symbol-Meaning Integration	10	3

Similarly, experts mentioned symbol-meaning integration 10 times, whereas novices did so only 3 times. Symbol-meaning integration reflects the ability to connect algebraic symbols with their underlying concepts or operations, a critical skill in abstract algebra. Experts' frequent emphasis on this subtheme indicates their superior capacity to interpret symbols contextually, facilitating deeper understanding and manipulation of abstract structures. Novices' fewer mentions suggest difficulties in linking symbolic notation to meaning, which can hinder comprehension and procedural fluency. This finding reinforces the cognitive load theory perspective that novices often face challenges managing the symbolic complexity of algebra (Sweller, 1988) and suggests that educators should focus on bridging the gap between symbols and their semantic content through targeted scaffolding. The disparity in subtheme frequencies between experts and novices reveals crucial cognitive differences in abstract algebraic reasoning. Experts' frequent references to mental templates and symbol-meaning integration reflect their advanced conceptual organization and symbol comprehension, which are less developed in novices.

For curriculum designers and educators, this highlights the need to: Incorporate explicit instruction and activities that help students build and apply mental templates, such as pattern recognition exercises and structured problem-solving frameworks. Enhance symbol-meaning integration through the use of multimodal teaching tools, including visual aids, verbal explanations, and interactive symbolic manipulations. By focusing on these areas, educators can better support novices' transition toward expert-like reasoning in abstract algebra.

4.3.2 Hierarchical Abstraction

Participants described moving between multiple levels of abstraction, often zooming in and out between concrete examples and generalized properties.

Subtheme 1: Contextual Anchoring
While working on abstract problems, participants anchored reasoning in specific instances, such as integer mod n examples, before generalizing to group axioms. A graduate student explained:

> "I start with an example, like integers mod 5, to get a concrete sense of how the structure behaves, and then I generalize to the group properties."

Subtheme 2: Recursive Abstraction
Participants demonstrated the ability to recursively abstract, moving from element-level considerations to set-level properties and back. This was especially pronounced in experts, who articulated this process explicitly.

Frequency Counts:

Subtheme	Expert Mentions	Novice Mentions
Contextual Anchoring	11	7
Recursive Abstraction	9	2

The subtheme of recursive abstraction shows a stark contrast: experts mentioned it 9 times, whereas novices mentioned it only 2 times. Recursive abstraction involves the ability to build successive layers of abstraction, reflecting a deepening of conceptual understanding through iterative generalization and reflection. Experts' frequent references highlight their advanced capacity to navigate complex hierarchies of algebraic structures by recursively abstracting properties and relations. This cognitive ability underpins the flexible manipulation of abstract entities, a hallmark of expertise in mathematics. The scarcity of recursive abstraction mentioned by novices indicates that they are less adept at this layered thinking, often remaining at more concrete or surface levels of reasoning. This limitation can hinder their progress toward deeper conceptual insights required for mastering abstract algebra.

The contrast in the frequencies of contextual anchoring and recursive abstraction between experts and novices illuminates important facets of cognitive development in abstract algebraic reasoning: Both groups recognize the importance of contextual anchoring, but experts employ it more strategically to scaffold and guide their reasoning. The pronounced difference in recursive abstraction points to a significant cognitive gap; developing recursive abstraction skills is critical for learners to achieve expert-level reasoning.

Instruction should emphasize helping students make meaningful connections to prior knowledge and real-world contexts to strengthen contextual anchoring. Teaching approaches that foster recursive abstraction—such as iterative problem-solving, meta-cognitive reflection, and encouraging learners to generalize from specific instances—can support novices in developing expert-like abstraction skills.

4.3.3 Procedural-Structural Duality

A recurring theme was the balance between procedural fluency (symbol manipulation) and structural understanding (conceptual insights).

Subtheme 1: Flexible Switching
Experts reported fluidly switching between algorithmic procedures and structural reasoning. One participant stated:

> "I don't just compute; I'm always asking why the computation works—what property or structure makes it valid."

Subtheme 2: Metacognitive Monitoring
Participants, particularly experts, engaged in metacognitive monitoring, consciously reflecting on whether their manipulations adhered to underlying algebraic principles. This self-regulation was less prevalent among novices, who often focused on procedural steps without considering structural implications.

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Frequency Counts:

Subtheme	Expert Mentions	Novice Mentions
Flexible Switching	13	5
Metacognitive Monitoring	10	3

Experts referenced flexible switching 13 times compared to 5 mentions by novices. Flexible switching refers to the ability to shift efficiently between different strategies, perspectives, or representations while engaging with abstract algebraic problems. This notable difference highlights a key characteristic of expert reasoning: adaptability. Experts demonstrate cognitive flexibility, moving seamlessly between algebraic structures, symbolic manipulations, conceptual understandings, and problem-solving strategies. Their frequent engagement in flexible switching suggests that they can adjust their approach based on problem demands, recognize multiple solution paths, and avoid fixation on a single method. In contrast, novices' limited mentions indicate a tendency toward rigid thinking and a struggle to adapt their strategies. This rigidity can result from underdeveloped conceptual frameworks and limited familiarity with algebraic reasoning, making them more prone to errors or inefficiencies when their initial approach is ineffective.

The frequency of metacognitive monitoring mentions also shows a sharp contrast: 10 times by experts versus 3 by novices. Metacognitive monitoring involves the conscious regulation of one's cognitive processes, such as planning, evaluating, and adjusting strategies based on progress and outcomes. Experts' higher mention rate suggests they actively and habitually reflect on their reasoning processes, assess the appropriateness of their approaches, and recognize when adjustments are necessary. This capacity to "think about one's thinking" is a hallmark of expert performance and is closely tied to successful problem-solving in complex, abstract domains like algebra. Novices' low frequency of mentions suggests limited metacognitive awareness. They may focus on surface-level operations without stepping back to assess their understanding or adjust their strategies. This limitation can impede learning, as it reduces opportunities to identify errors and refine problem-solving approaches.

These findings illustrate important differences in the cognitive strategies of experts and novices in abstract algebraic reasoning: Flexible Switching is more prevalent among experts, highlighting their adaptability and broad problem-solving repertoire. Metacognitive Monitoring is also more frequent among experts, underscoring the importance of reflective and self-regulatory processes in navigating complex tasks.

To help novices develop flexible switching, educators can introduce varied problem types, encourage multiple solution approaches, and model switching strategies explicitly. For metacognitive development, structured reflective activities (e.g., think-aloud protocols, problem-solving journals, and self-assessment checklists) can scaffold learners' ability to monitor and regulate their reasoning. By fostering these cognitive skills, instructional interventions can help bridge the gap between novice and expert reasoning in abstract algebra.

4.3.4 Emergent Subthemes

During coding, two emergent subthemes were identified that cut across the primary themes:

Epistemic Humility: Some participants expressed awareness of the limitations of procedural reasoning, emphasizing a need for structural insight:

> "When I hit a wall procedurally, I step back and think about the bigger structure—it's humbling but necessary."

Collaborative Visualization: A subset of participants described collaborative problem-solving as a means to externalize cognitive structures, such as drawing commutative diagrams or lattice representations, to scaffold reasoning.

Frequency Counts:

Emergent Subtheme	Expert Mentions	Novice Mentions
Epistemic Humility	8	2
Collaborative Visualization	7	3

Experts mentioned epistemic humility 8 times, compared to just 2 mentions by novices. Epistemic humility refers to the recognition of the limits of one's knowledge and a willingness to revise beliefs or approaches based on new evidence or alternative viewpoints. Experts' higher frequency of this subtheme highlights a key characteristic of expert reasoning: openness to uncertainty and adaptability. Their readiness to acknowledge gaps in their understanding or adjust their strategies likely reflects a mature, reflective approach to complex problem-solving. This mindset fosters continuous learning and innovation, especially in abstract fields like algebra where problems can often have multiple pathways and interpretations. Novices' low mention frequency may reflect a lack of awareness or confidence in recognizing knowledge limitations. Novices may either overestimate their understanding or hesitate to express uncertainty, which can impede their learning and adaptation.

Collaborative visualization was mentioned 7 times by experts and 3 times by novices. This subtheme involves the use of shared visual representations (e.g., diagrams, sketches, or symbolic representations) to facilitate problem-solving in a group setting. Experts' more frequent references suggest that they recognize and utilize visual tools to co-construct understanding and support collective reasoning. Their ability to externalize and negotiate abstract concepts through collaborative visual aids contributes to clearer communication, deeper engagement, and more effective joint problem-solving. Novices, while less frequent in their mentions, do acknowledge the utility of visualization but may not yet fully appreciate its potential as a collaborative tool. They might rely more on individual reasoning or struggle to integrate visual representations effectively in group contexts.

These emergent subthemes highlight not only cognitive but also social and affective dimensions of expert reasoning: Epistemic humility among experts fosters adaptive learning, critical reflection, and openness to new ideas—all essential for mastering abstract algebra. Collaborative visualization supports collective cognitive processes, enabling clearer articulation of complex concepts and fostering group problem-solving skills.

Encourage epistemic humility by creating a classroom culture where uncertainty is normalized, and learners are encouraged to question, reflect, and revise. Techniques include open-ended problems, peer review, and reflection prompts. Promote collaborative visualization through group-based activities like

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whiteboarding, collaborative mind-mapping, and dynamic visual tools (e.g., digital math platforms). These practices can help novices develop both visual reasoning skills and collaborative problem-solving abilities.

5. Discussion

The findings support the existence of specialized cognitive frameworks in abstract algebraic reasoning. The neurocognitive evidence aligns with the hypothesis of domain-specific modules facilitating symbolic manipulation. Eye-tracking data highlight the role of attentional strategies in managing cognitive load. The thematic insights emphasize the dynamic interplay between procedural and structural reasoning, suggesting a model of algebraic cognition that integrates symbolic schemas, hierarchical abstraction, and dual processing.

The integration of neurocognitive, eye-tracking, and qualitative data provides a comprehensive view of the cognitive structures underlying abstract algebraic reasoning. The activation of the DLPFC and IPS aligns with their roles in working memory and symbolic processing, supporting the hypothesis of domain-specific cognitive modules. Eye-tracking patterns suggest that experts allocate visual attention more efficiently, focusing on structural elements critical to problem-solving. The thematic analysis underscores the importance of internalized symbolic schemas and the interplay between procedural and structural understanding. These findings have implications for educational practices, suggesting that fostering structural awareness and symbolic schema development may enhance proficiency in abstract algebra.

The thematic findings deepen our understanding of how cognitive structures in abstract algebra are constructed and navigated. The prominence of symbolic schemas suggests that expertise involves the internalization of domain-specific representations, consistent with chunking theory (Chi et al., 1982). Hierarchical abstraction supports the notion of recursive reasoning levels, echoing Sfard's (1991) duality of process and object. The procedural-structural duality theme highlights the importance of integrating procedural fluency with structural understanding, aligning with Tall's (2004) concept of the proceptual nature of mathematical symbols.

Moreover, epistemic humility and collaborative visualization provide novel insights, suggesting avenues for pedagogical interventions. By explicitly teaching students to step back from procedural walls and externalize cognitive structures through diagrams, educators may cultivate deeper algebraic reasoning skills.

6. Conclusion

This study advances our understanding of the cognitive structures underlying abstract algebra. The identification of domain-specific modules and reasoning strategies opens pathways for enhancing algebra education and developing targeted cognitive interventions. Future research should expand on these findings using longitudinal designs and diverse participant populations.

This study advances our understanding of the cognitive structures underlying abstract algebra. The identification of domain-specific modules and reasoning strategies opens pathways for enhancing algebra education and developing targeted cognitive interventions. Future research should expand on these findings using longitudinal designs and diverse participant populations.

7. Recommendations

Based on the integrated findings from thematic analysis, neurocognitive data and eye-tracking metrics, several recommendations emerge to enhance the teaching and learning of abstract algebraic reasoning:

7.1 Foster Symbolic Schema Development

Educational practices should prioritize the explicit teaching of symbolic schemas. Instructors can: Introduce canonical structures (e.g., cyclic groups, rings) early in the curriculum. Encourage students to verbalize and diagram the relationships between symbols and their meanings. Use worked examples and contrasting cases to highlight how symbolic templates can guide problem-solving.

7.2 Promote Hierarchical Abstraction Skills

To cultivate students' ability to move between levels of abstraction: Incorporate tasks that require transitioning from concrete examples to generalized concepts. Use scaffolding tools (e.g., number lines, modular arithmetic examples) to anchor understanding. Encourage metacognitive reflection on how concrete cases inform abstract reasoning.

7.3 Balance Procedural Fluency and Structural Understanding

Instructional strategies should balance computational practice with structural insight: Design tasks that explicitly link procedures to properties (e.g., asking "Why does this procedure work?"). Implement problem-based learning that requires both symbolic manipulation and conceptual justification. Incorporate collaborative activities where students must explain their reasoning to peers.

7.4 Support Epistemic Humility and Visualization

Recognizing the role of epistemic humility and collaborative visualization: Create a classroom culture that normalizes acknowledging gaps in understanding and seeking structural explanations. Use visualization tools such as lattice diagrams, group tables, and concept maps to externalize reasoning. Facilitate group work and discussions, enabling collaborative construction of visual representations.

7.5 Implications for Curriculum Design

Curriculum designers should: Integrate visual and conceptual learning tools into abstract algebra courses. Sequence topics to build from concrete to abstract, mirroring the natural progression of cognitive development identified in the study. Include assessments that measure both procedural accuracy and structural insight, ensuring a comprehensive evaluation of reasoning skills.

7.6 Directions for Future Research

Future studies could: Expand participant diversity, including undergraduates, advanced learners, and mathematicians from varied cultural contexts. Investigate longitudinal development of cognitive structures in abstract algebra, tracking changes over time. Explore the role of digital tools and AI, such as intelligent tutoring systems, in supporting algebraic reasoning. Combine neuroimaging with real-time task analysis to capture dynamic cognitive processes during problem-solving.

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